The Elusive Perfect Problem

“Do No Harm” activities in an enrichment program for “unenriched” students

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San Francisco Math Circle

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SF Math Circle: Joint Work

- Started Fall 2005

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- Instructors, Teachers, Students, Parents

What Type of Circle?

- Students: Unenriched —— Already enriched
- Diversity: High —— Low
- Recruitment: Teacher — Self — Parents
- Time: After-school —— Evening/Weekend
- Length: Short —— Long
- Level: Math doesn't suck! —— Olympiad
- Instruction: Small groups —— Pure Lecture

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- Easier, stand-alone

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- Good: What the students invented (dots)

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- The smaller of these two factors equals n; the larger equals $a + \ell$.

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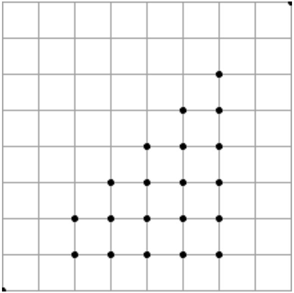
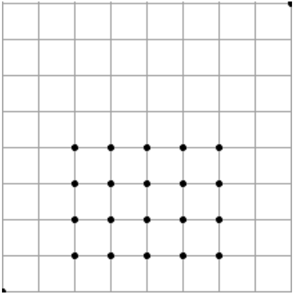
  - $T = 4 + 5 + 6 + 7.$

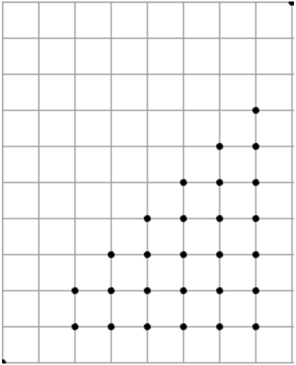
Dots to the Rescue! $2 + 3 + 4 + 5 + 6 = ?$



$2 + 3 + 4 + 5 + 6 = 5 \cdot 4$



$2 + 3 + 4 + 5 + 6 + 7 =?$



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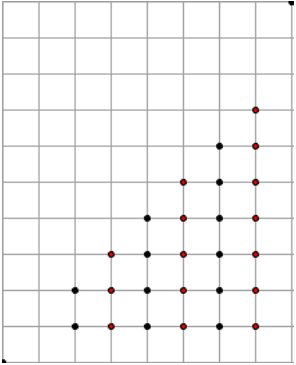

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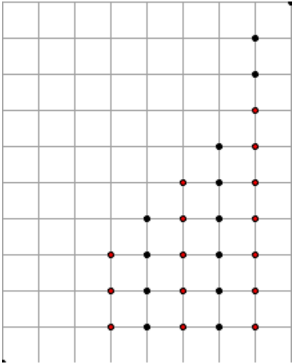

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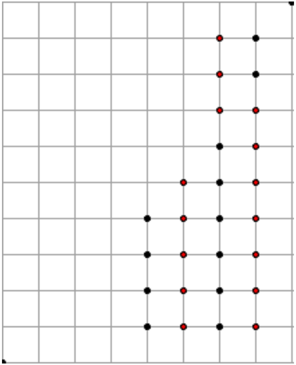
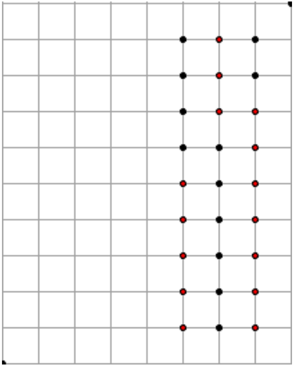


$2 + 3 + 4 + 5 + 6 + 7 = 3 \cdot 9$



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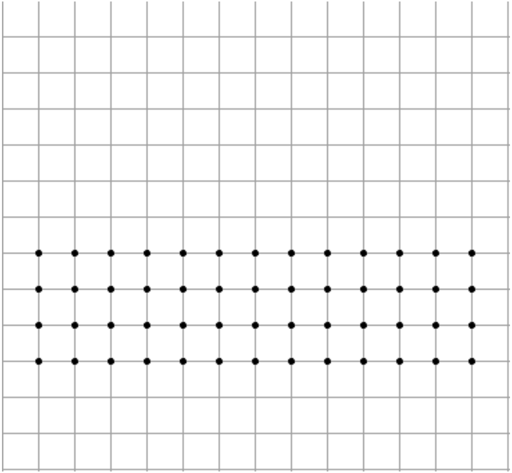

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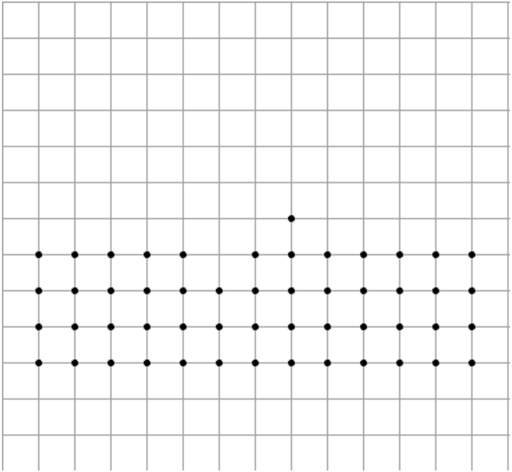

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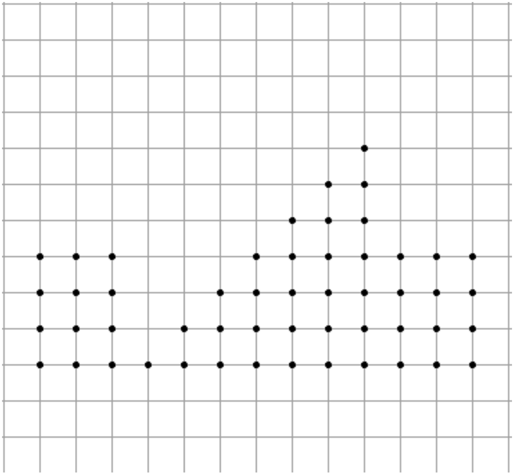

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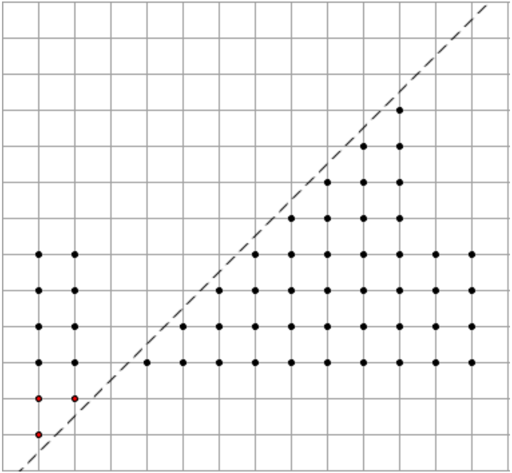

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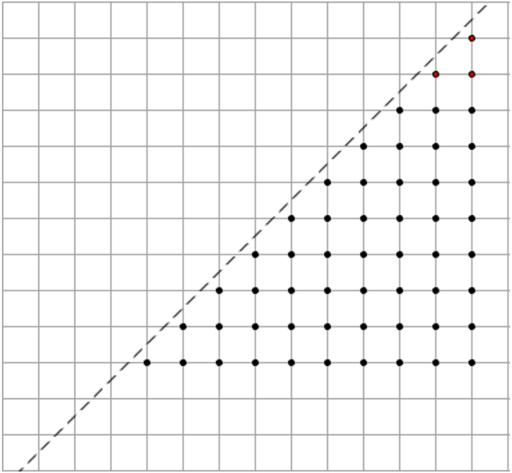
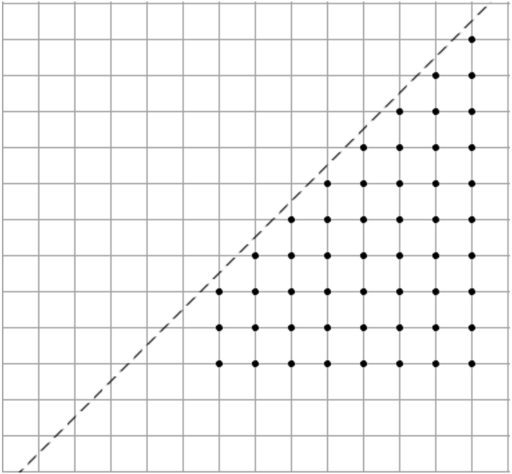


$13 \times 4 = 3 + 4 + \cdots + 9 + 10$



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  - Opaque cards are labeled $1/2$, $2/3$, or $3/4$.
  - Two opposing players sit opposite one another
  - Moderator holds up a card so that each player sees one side.
  - First player to say which number her opponent sees wins.

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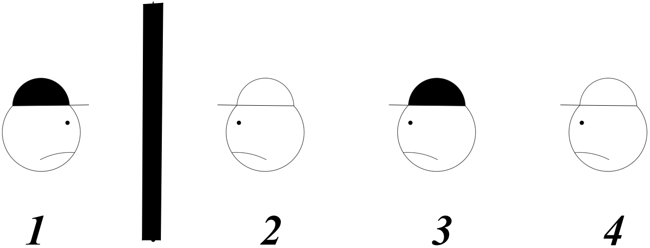
- Warmup problem II : Heads in the Sand



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- Starting from the rear, each person will say what color their hat is. The moderator will tell them if they are right or wrong. They are ONLY allowed to say "black" or "white."

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Assume that they can meet for a strategy meeting before the hats are put on. How can they maximize the number of correct answers?

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- Trained student helpers/performers