KenKen as a Mathematical Object

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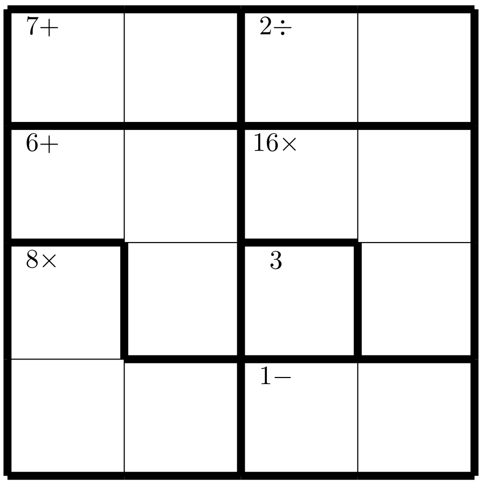
KenKen® is a puzzle whose solution requires a combination of logical, simple arithmetic and combinatorial skills. The puzzles range in difficulty from very simple to incredibly difficult. Students who get hooked on the puzzle will find themselves practicing addition, subtraction, multiplication and division facts. This paper is intended for mathematicians who want to use KenKen puzzles to nurture those skills in students. It is also an attempt to build a mathematical framework that enables the generalization of KenKen to puzzles for which the arithmetic is different from our usual one.

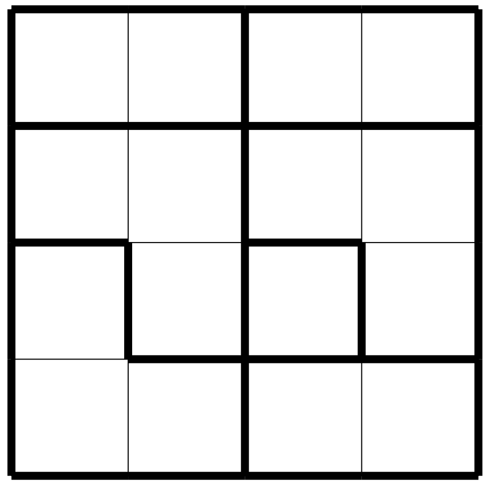
Websites of interest:

1. `http://www.math.uncc.edu/~hbreiter/KenKen`
2. `http://www.geometer.org/mathcircles/kenken.pdf`
3. `http://www.kenken.com/`
4. `http://www.nytimes.com/ref/crosswords/kenken.html`
5. `http://www.stanford.edu/~tsnyder/kenken.htm`
6. `http://en.wikipedia.org/wiki/KenKen`

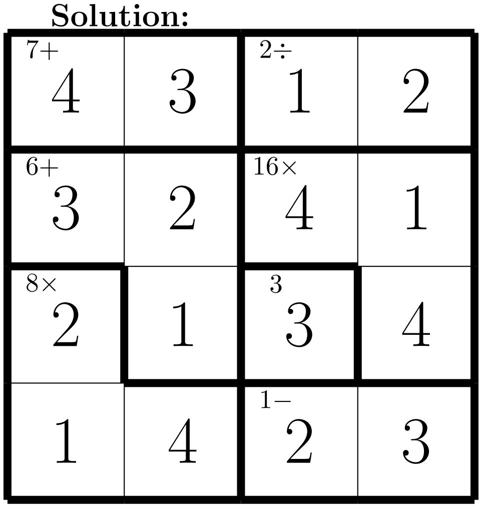
1 A Sample Puzzle

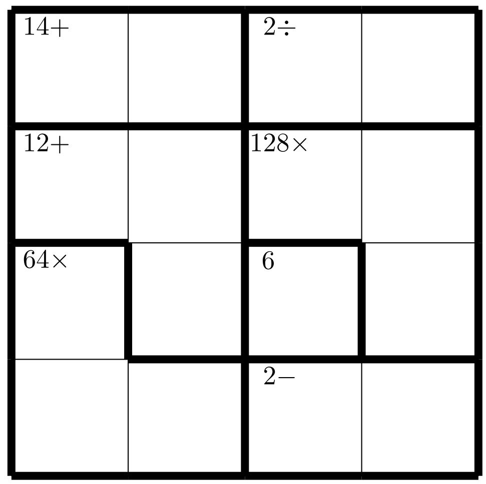
Look at the two 4×4 puzzles below. Solve the first one and then find clues for the second one so that a distribution of the alphabet $\{2, 4, 6, 8\}$ uniquely solve the puzzle.





Solution:





2 What is KenKen?

Let $\mathcal{A}$ be an alphabet of n symbols, and let $\otimes$ and $\oplus$ be associative, commutative binary operators on $\mathcal{A}$. A KenKen puzzle is a four-tuple $K = (\mathcal{A}, \mathbb{C}, \otimes, \oplus)$, where $\mathbb{C} = \{C_1, C_2, \ldots, C_k\}$ is a partition of an $n \times n$ checkerboard into polygonal regions (sometimes called polyominoes) each of which is the union of unit squares (also called cells), and each of which comes with a clue c_i of the following type. In case C_i consists of three or more squares, the clue c_i is one of

- ϕ, the empty clue,
- $x\otimes$, where $x \in \mathcal{A}$, or
- $x\oplus$, where $x \in \mathcal{A}$.

A multisubset $x_1, x_2, \ldots, x_t$ of $\mathcal{A}$ satisfies a t-cell cage clue $x\oplus$ provided $x_1 \oplus x_2 \oplus \ldots \oplus x_t = x$. A multisubset $x_1, x_2, \ldots, x_t$ of $\mathcal{A}$ satisfies a t-cell cage clue $x\otimes$ provided $x_1 \otimes x_2 \otimes \ldots \otimes x_t = x$. Every multisubset satisfies the empty clue.

In case C_i consists of exactly two squares, the clue c_i can be any of the three above or either of the two below.

- $x\ominus$, where $x \in \mathcal{A}$, or
- $x\oslash$, where $x \in \mathcal{A}$.

In case C_i is a single square, c_i is either the empty clue of a member of $\mathcal{A}$.

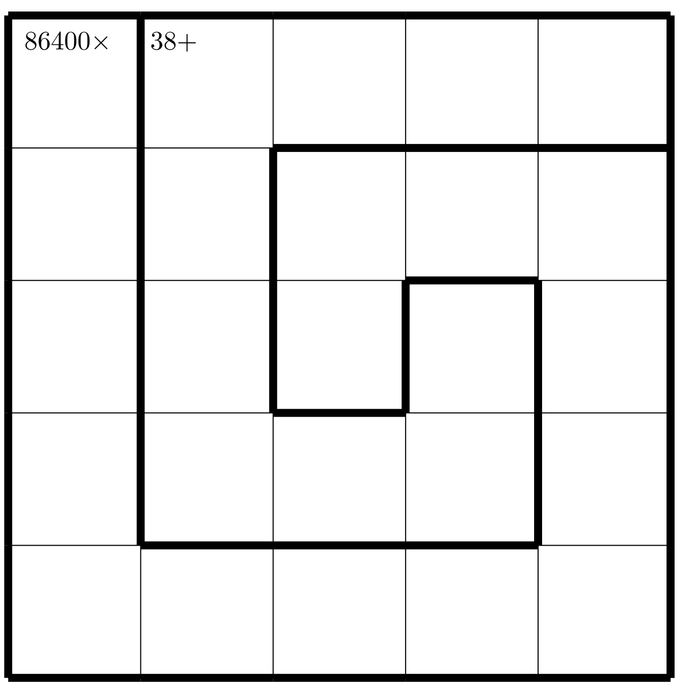
A pair u, v of elements of $\mathcal{A}$ *satisfies* a clue $x\ominus$ if either $u \oplus x = v$ or $v \oplus x = u$. A pair u, v of elements of $\mathcal{A}$ *satisfies* a clue $x\oslash$ if either $u \otimes x = v$ or $v \otimes x = u$. In case C_i is a single square with clue x, only x itself satisfies the clue. If the clue is empty, all the elements of $\mathcal{A}$ satisfy the clue.

A *solution* to a KenKen puzzle is an $n \times n$ matrix of elements of $\mathcal{A}$ such that no two members of $\mathcal{A}$ appear in any row or column and such that each clue is satisfied. A KenKen puzzle is called *standard* if there is exactly one such matrix. **Standard KenKen** (see `http://www.kenken.com/`) in an $n \times n$ puzzle with the alphabet $1, 2, 3, \ldots, n$, no empty cages, and the usual operators $+, \times, -$, and $\div$.

Some puzzle creators (see `http://www.stanford.edu/~tsnyder/kenken.htm`, for example) use an ordered alphabet, and define cages of more than 2 cells with clues $\ominus$, and $\oslash$ as follows. The value of a multiset is the largest of the cage $\ominus$ or $\oslash$ combined with the result of applying $\oplus$ or *otimes* to the rest of the members of the multiset. For example $\ominus$ applied to the multiset $\{1, 2, 6\}$ gives the result $6 - (1_2) = 3$.

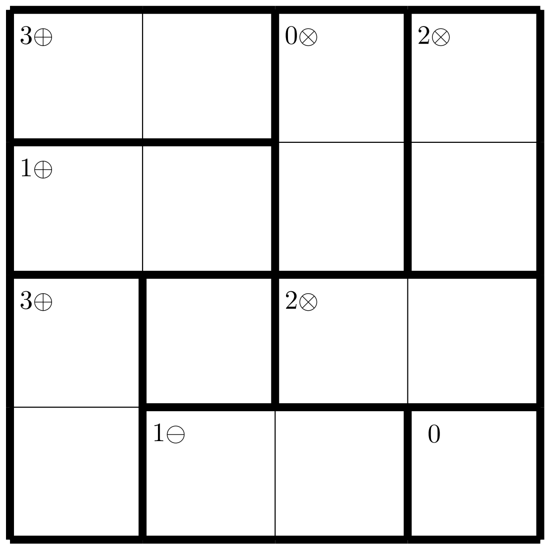
3 An Example

In the example below, $\mathcal{A} = \{1, 2, 3, 4, 5\}$ and $\mathbb{C}$ consists of exactly two cages. The operations $\oplus$ and $\otimes$ are just the usual $+$ and $\times$.



Solution: The easy way to approach this is to note that in the cage [86400×] there must be exactly two 5's and exactly three 3's. Thus the [38+] cage must have three 5's and two 3's among its 10 digits. Therefore, the sum of the other five digits is $38 - 21 = 17$, and the only way to achieve this is four 4's and a 1. Now it turns out that there is just one way to put four 4's, three 5's, two 3's, and one 1 in the cage, and once this is done, the other cage can be filled in only one way.

4 Example 2

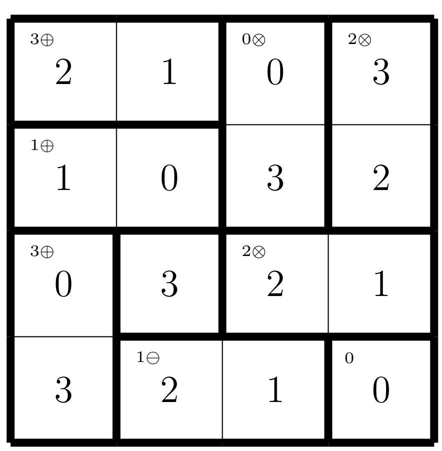


Addition $\oplus$ and multiplication $\otimes$ on $\mathcal{A} = \{0, 1, 2, 3\}$

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\odot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

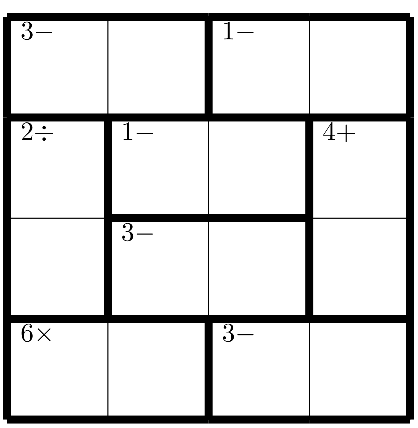
The complete solution is given below.

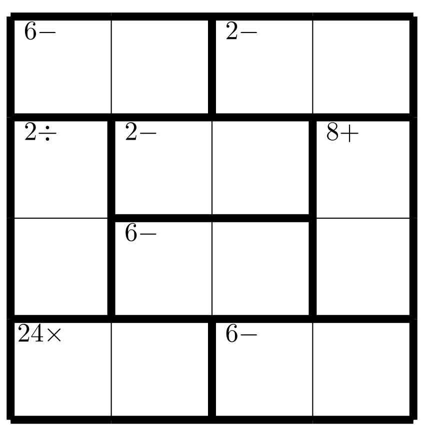


To fully understand what a KenKen puzzle is, we need some more terminology. If C is a t-cell cage with clue c, a *candidate set* for c is a t-element multiset T_c that satisfies the clue c. Of course, most cages have more than one candidate set. For example, in a 6×6 standard KenKen, a two-cell cage with clue 7+ has three candidate sets $\{1, 6\}, \{2, 5\}$, and $\{3, 4\}$. Note that the shape of the cage can influence the candidate set. For example, in a 6×6 standard KenKen, the 3-cell L-shaped cage with clue 6+ is $\{1, 1, 4\}, \{1, 2, 3\}$, but the straight 3-cell cage with the same clue has only one candidate set, $\{1, 2, 3\}$.

5 Isomorphism

In this section we explore the notion of isomorphism. Suppose we have two $n \times n$ KenKen puzzles $K_1 = (\mathcal{A}_1, \mathbb{C}_1, \otimes_1, \oplus_1)$ and $K_2 = (\mathcal{A}_2, \mathbb{C}_2, \otimes_2, \oplus_2)$. We call K_1 and K_2 isomorphic, and write $K_1 \cong K_2$ if there are two bijective functions $f : A_1 \to A_2$ and $g : \mathbb{C}_1 \to \mathbb{C}_2$ satisfying the following condition: for each cage $C_1 \in \mathbb{C}_1$, the multiset T_{C_1} is a candidate set for C_1 if and only if $f(T_{C_1})$ is a candidate set for $g(C_1)$. Below is an example of two isomorphic puzzles. The 4×4 puzzle K_1 on the left has alphabet $\{1, 2, 3, 4\}$ while K_2 on the right has alphabet $\{2, 4, 6, 8\}$. The function $f : A_1 \to A_2$ is given by $f(n) = 2n$. Its pretty clear how the function g is defined.





Theorem 1 $\cong$ is an equivalence relation on the family of all $n \times n$ KenKen puzzles. That is, $\cong$ is reflexive ($K \cong K$, $\forall K$), symmetric ($K \cong L \Longrightarrow L \cong K$), and transitive ($K \cong L \wedge L \cong M \Longrightarrow K \cong M$). The proof is left to the reader.

6 Measuring Difficulty

What makes a good puzzle? Most puzzlers believe that a great puzzle must not require slogging and must reward insight. A great puzzle must not give up after the first insight. It must keep fighting back. Of course, these are not mathematical ideas, and so they are hard to quantify. Yet there is a way to say one puzzle is harder than another. Suppose in a 6×6 standard puzzle, a [15×] cage is replaced by a [8+] cage and that the later still has a unique solution. Then clearly the later is harder than the former. We can use the terminology developed in the previous section to formalize this idea.

Suppose we have two $n \times n$ KenKen puzzles $K_1 = (\mathcal{A}_1, \mathbb{C}_1, \otimes_1, \oplus_1)$ and $K_2 = (\mathcal{A}_2, \mathbb{C}_2, \otimes_2, \oplus_2)$. Suppose there are a bijective functions $f : A_1 \to A_2$ and $g : \mathbb{C}_1 \to \mathbb{C}_2$ satisfying the following condition: for each cage $C_1 \in \mathbb{C}_1$, if multiset T_{C_1} is a candidate set for C_1 then $f(T_{C_1})$ is a candidate set for $g(C_1)$. In this case we say K_1 is *easier* than K_2 or K_2 is *harder* than K_1 and write $K_1 \prec K_2$.

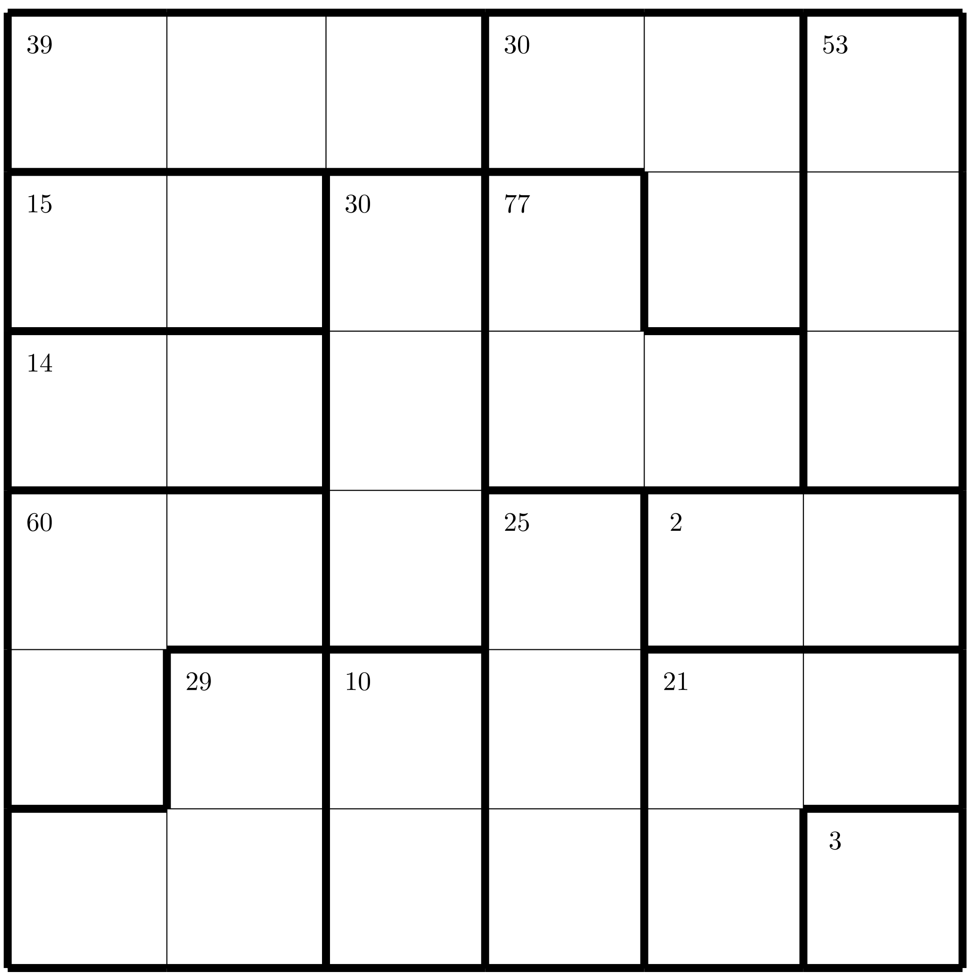
Now its not very hard to prove

Theorem 2 $\prec$ is a partial ordering of the family of all $n \times n$ KenKen puzzles. That is, *prec* is reflexive ($K \prec K$, $\forall K$), antisymmetric ($K \prec L \wedge L \prec K \Longrightarrow K \cong L$, and transitive ($K \prec L \wedge L \prec M \Longrightarrow K \prec M$). The proof is left to the reader.

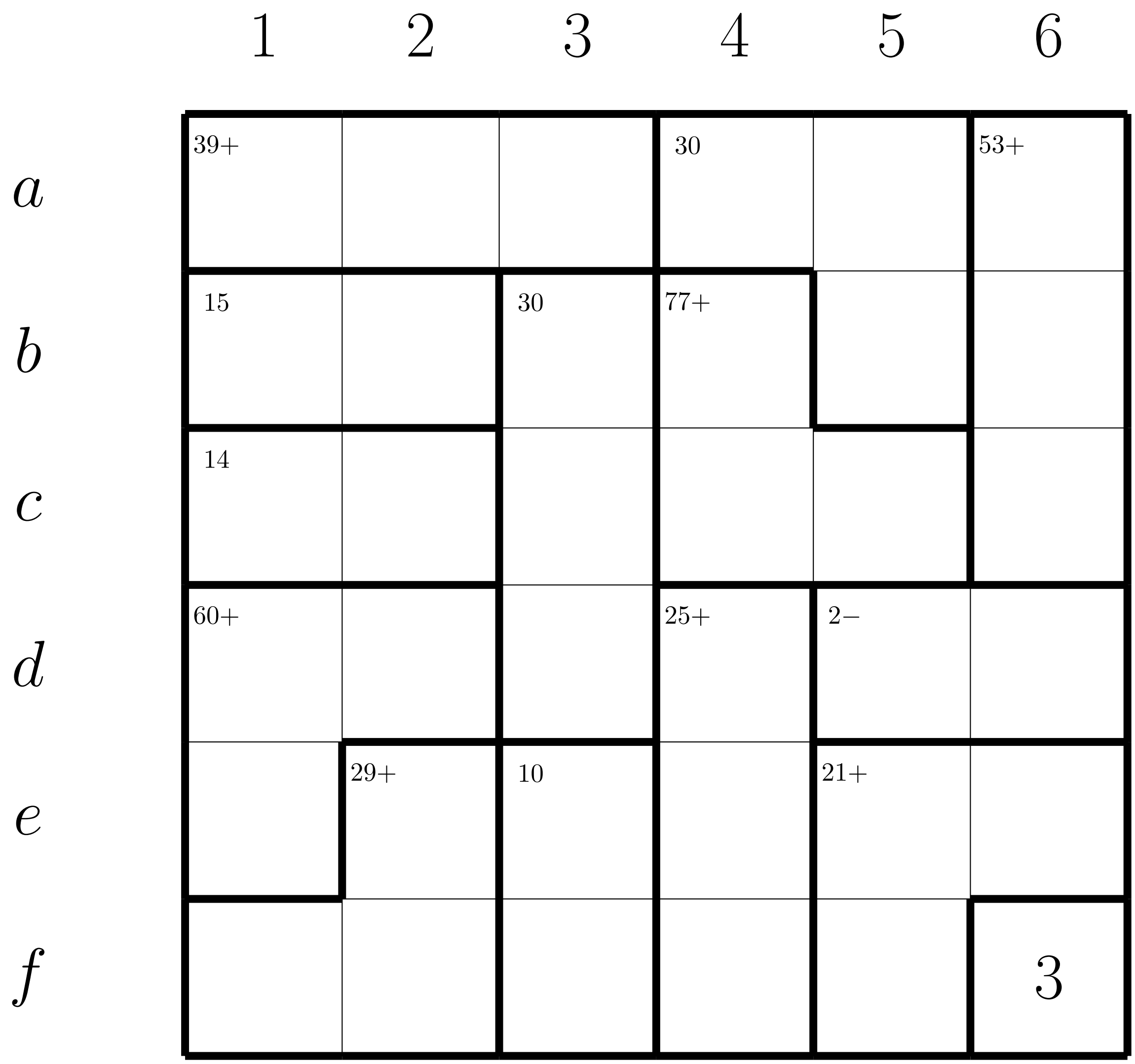
We give an example below. Of course, removing a clue from a cage is one way to make a puzzle harder. The problem is that if we push the puzzle to become harder, we may get a puzzle with multiple solutions.

7 Prime KenKen

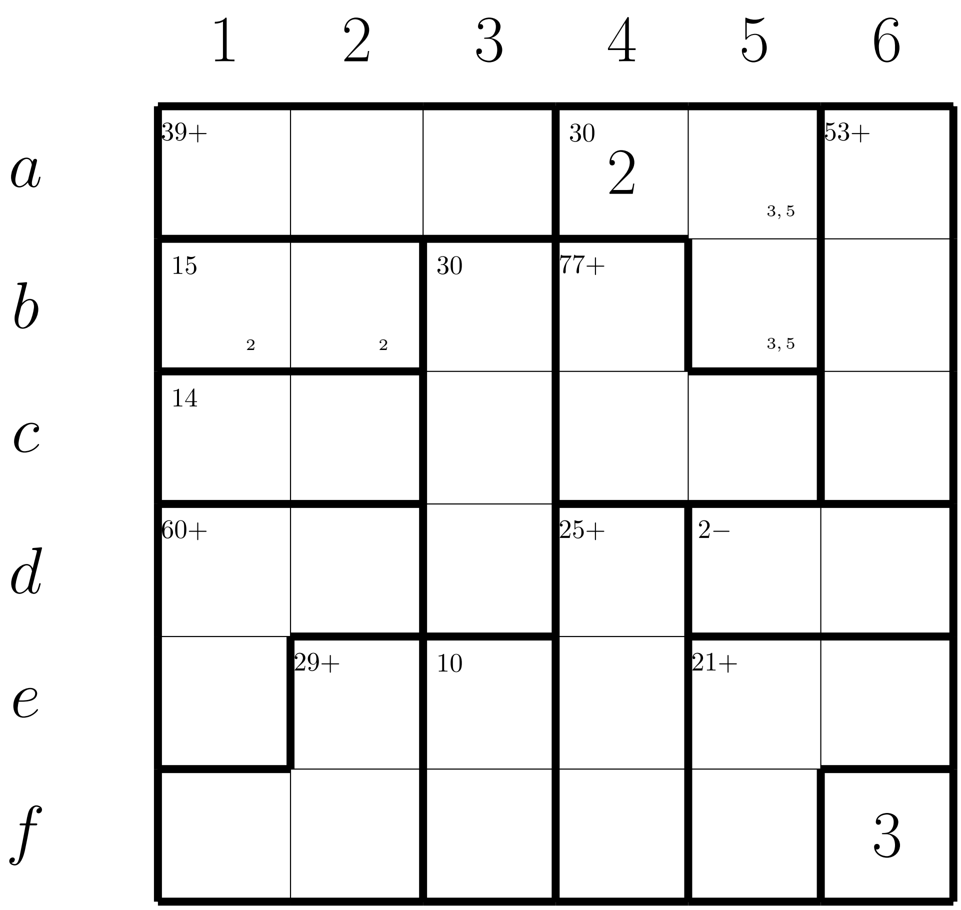
In the regular puzzle KenKen, the numbers in each heavily outlined set of squares, called *cages*, must combine (in any order) to produce the *target number* in the top corner of the cage using the mathematical operation indicated. A number can be repeated within a cage as long as it is not in the same row or column. In this 6×6 puzzle, the **six numbers** are known only to be prime numbers. In contrast to most KenKen puzzles, here you must figure out which operations produce the target numbers. Of course any cage with more than two cells must be multiplication or addition.



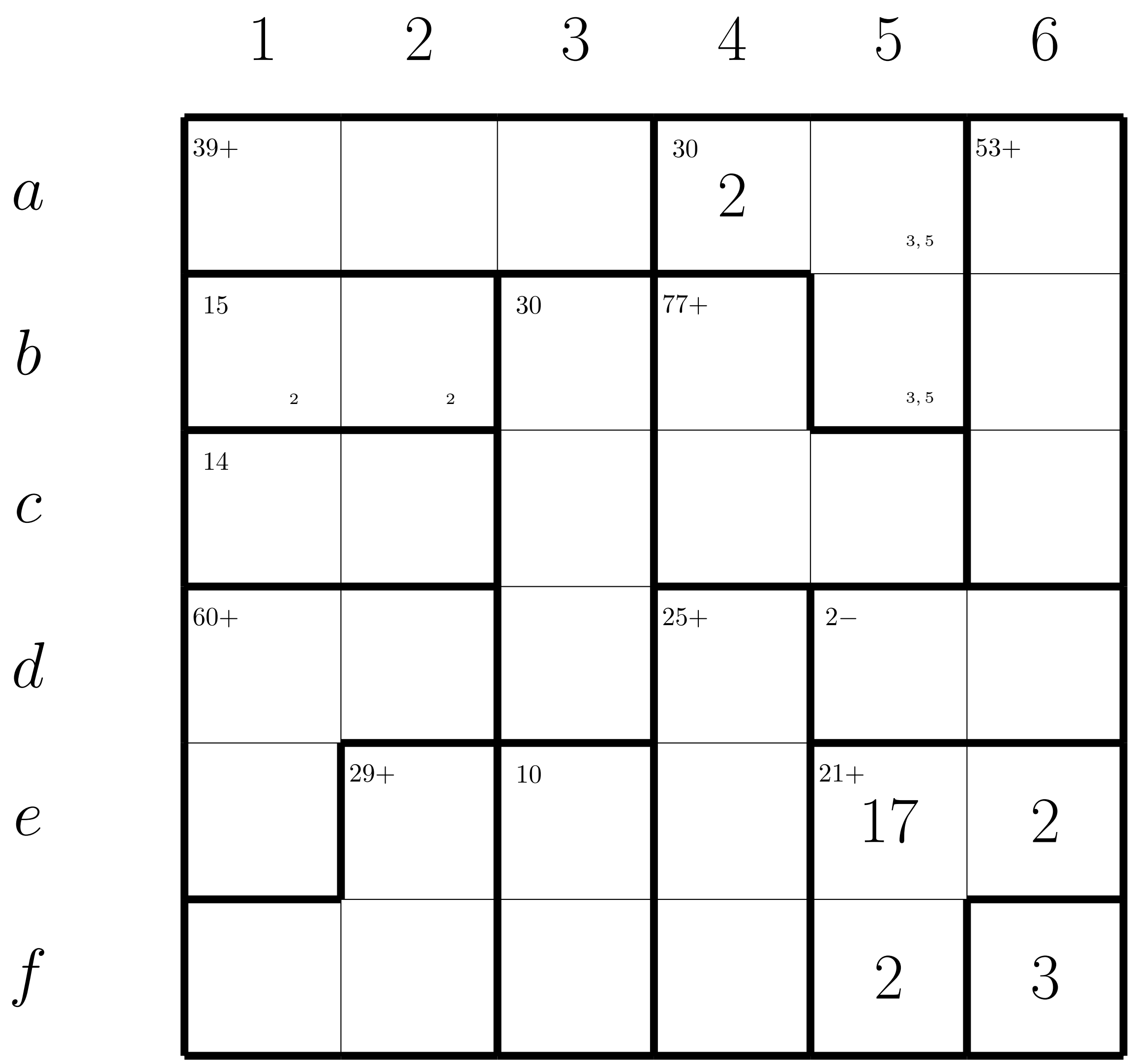
Solution: First, let's label all the clues that we can with the operation. Let's also name the rows and columns so that we can quickly identify each location. Thus,



Solution: As before, let $T = \{x, y, z, u, v, w\}$ be the set of 6 primes, and σ their sum. Now $2 \in T$ because the cage [60+] must have a 2. Next note that neither of the cages [25+] and [77+] can have a 2 because they are 3-cell cages with an odd sum. Therefore the 2 of column 4 goes in $a4$. Next note that the sum of the entries in the columns 4, 5 and 6 is 3σ, a multiple of 3. Thus $53+77+25+21+3+\sum[30]+\sum[2-] = 179+\sum[30]+\sum[2-]$ is a multiple of 3. Let's rule out $\{[2-]\} = \{3, 5\}$. If $\{[2-]\} = \{3, 5\}$, then $\sum[2-] = 8$. Since $[30] = [30+]$ or $[30\times]$, it follows that $\sum[30] = 2+3+5 = 10$ or 30. But neither $179+10+8$ or $179+30+8$ are multiples of 3. Now all the other possibilities for the [2−] cage have the property that $\sum[2-]$ is a multiple of 3. Why? This implies that $[30] = [30\times]$ since $179+10 = 189$ is a multiple of 3 while $179 + 30 = 209$ is not. Thus $\{a5, b5\} = \{3, 5\}$. What does this imply about the cage [15] at $b2$? The answer is that [15] cannot be [15×] because either a 3 or a 5 must occupy $b5$. Now this means that either $\{[15]\} = \{[15+]\} = \{13, 2\}$ or $\{[15]\} = \{[15-]\} = \{17, 2\}$. Let's summarize what we know in the grid below.



Now, where's the 2 in column 6? Its not in the [53+] cage, and its not in the [2−] cage, so it must be at $e6$. This requires yet another 2 in the cage [21+]. So we have



Now let's find the 2 in the other [30] cage. It can't be in row d because the 2 in the [60+] cage must be in row d. But there is also a 2 in row b in the [15] cage. So we have a 2 at $c3$. This implies that the [14] cage cannot be [14×], hence must be [14−] or [14+]. What we know now is that $T = \{2, 3, 5, 17, v, w\}$. We are almost done. If the numbers u and $u + 2$ that occupy the [2−] cage are v and w, then we can write $3\sigma = 189 + 2u + 2 = 3(2 + 3 + 5 + 17 + u + u + 2)$, and this results in a value of $u = 26$, a contradiction. Therefore $u = 5$ or $u = 17$. The former results in $u + 2 = 7$ and $T = \{2, 3, 5, 7, 17, w\}$, where $w = 33$, a contradiction. If $u = 17$, then $u + 2 = 19$ and we have $T = \{2, 3, 5, 17, 19, 29\}$ and $\sigma = (189 + 36) \div 3 = 75$, which is consistent. The complete solution is given below.

	1	2	3	4	5	6
a	39+ 19	3	17	30 2	5	53+ 29
b	15 17	2	30 5	77+ 29	3	19
c	14 3	17	2	19	29	5
d	60+ 2	29	3	25+ 5	2− 19	17
e	29	29+ 5	10 19	3	21+ 17	2
f	5	19	29	17	2	3